

# The $b$ -boundary of Lorentzian surfaces

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The  $b$ -boundary is a mathematical tool able to attach a topological boundary to incomplete Lorentzian manifolds. This has been a motivation for using the  $b$ -boundary to study gravitational singularities in General Relativity. In this paper we give the general technique and mathematical preliminaries needed for the construction of such boundary. Furthermore, we explore the geometry of  $1+1$  spacetimes and give several examples.

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## I. INTRODUCTION

One of the biggest surprises that General Relativity (GR) has given us is that under certain circumstances the theory predicts its own limitations. There are two physical situations where we expect the theory to break down. The first one is the gravitational collapse of certain massive stars when their nuclear fuel is spent. The second one is the far past of the universe when the density and temperature were extreme. In both cases, we expect that the geometry of spacetime will show some pathological behaviour or singularities.

The nature of a gravitational singularity is a delicate issue. It might be tempting to define a gravitational singularity following other physical theories (such as electromagnetism) as the location where the relevant physical quantities are undefined. However, in the gravitational case this prescription does not work due to the identification of the spacetime background with the gravitational field. Hence, the concepts of ‘spatial location’ and ‘temporal duration’ have meaning only in such places where the gravitational field is defined. This represents a problem because the size, place and shape of singularities can not be straightforwardly characterised by any physical measurement.

The first formal notion of a gravitational singularity comes from Penrose and Hawking in their seminal theorems. They managed to show that GR breaks down under certain conditions [1]. Broadly speaking, the theorems establish that a spacetime  $(\mathcal{M}, g)$  that satisfies simultaneously

- a condition on the curvature,
- an appropriate initial or boundary condition,
- and a global causal condition,

must be geodesically incomplete [2].

The above mentioned theorems implicitly assume a characterisation of singularities in terms of geodesic incompleteness. This notion of singularity captures the idea that there are ‘obstructions’ within the history of point-like free falling observers. One would like extending those theorems to relate them to curvature blow-up and obstructions to the well-possessedness of initial value problems of field equations. This approach constitutes most of the research program on the Strong Cosmic Censorship conjecture [3] [4], the idea behind generalised hyperbolicity [5] [6] [7] [8] and field regularity [9] [10] [11] [12] [13] [14]. Other approaches have focused in providing the incomplete Lorentzian manifold with a boundary.

The procedure to attach a boundary to a Lorentzian manifold can be done in several nonequivalent ways. In this work we will focus on the  $b$ -boundary method [15]. This method allows a classification of singularities in terms of parallel propagated frames, it distinguishes between points at infinity and points at a finite distance, and it generalises the idea of affine length to all curves regardless of them being geodesic or not. Other common techniques to attach boundaries to Lorentzian manifolds are conformal boundaries [1] [16], causal boundaries [1] and approaches that mix several techniques like the  $a$ -boundary [17]. We give now a brief description of such techniques.

The conformal boundary allows us to study the structure of the metric at “infinity”. The idea of conformal compactification is to bring points at “infinity” on a non-compact pseudo-Riemannian manifold  $(\mathcal{M}, g)$  to a finite distance (in a new metric) by a conformal rescaling of the metric  $\tilde{g} = \Omega^2 g$ . This precise definition of conformal compactification only applies to an asymptotically simple spacetime.  $(\mathcal{M}, g)$  is asymptotically simple, if there are another smooth Lorentz manifold and associated metric  $(\tilde{\mathcal{M}}, \tilde{g})$  such that:

- $\mathcal{M}$  is an open sub-manifold of  $\tilde{\mathcal{M}}$  with smooth boundary  $\partial\mathcal{M}$  called the conformal boundary;
- there exists a smooth scalar field  $\Omega$  on  $\tilde{\mathcal{M}}$  such that  $\tilde{g} = \Omega^2 g$  on  $\mathcal{M}$ , and  $\Omega = 0, d\Omega \neq 0$  on  $\partial\mathcal{M}$ ;
- every null geodesic in  $\mathcal{M}$  acquires a future and past endpoints on  $\partial\mathcal{M}$ .

This technique has the evident drawback that it can only be applied to this kind of spacetimes [16]. Moreover, notice that in Minkowski spacetime the conformal boundary is given by  $\partial\mathcal{M} = \mathcal{I}^- \cup \mathcal{I}^+$  (where  $\mathcal{I}^-$  corresponds to null-past infinity and  $\mathcal{I}^+$  corresponds to null-future infinity) while  $i^o, i^+, i^-$  which correspond to spacelike infinity, future timelike infinity, and past timelike infinity respectively do not belong to the conformal boundary (the thorough reader can find in [1] formal definitions of  $\mathcal{I}^-, \mathcal{I}^+, i^o, i^+, i^-$ ). The reason for this is because  $\partial\mathcal{M}$  is not a smooth manifold at these points. Despite this, the conformal boundary has been successfully applied to study isolated systems in General Relativity [16] and more recently to the AdS-CFT correspondence [18].

Giving a causal boundary to a spacetime consists on attaching a boundary that depends only on the causal structure. However, this implies on this particular construction that one is not able to distinguish between boundary points and points at infinity. Moreover one has to assume that  $(\mathcal{M}, g)$  is strongly causal. This construction relies on indecomposable past sets (IP) and indecomposable future sets (IF) which we now define. An open set  $U$  is an IP if it satisfies  $I^-(U) \subset U$  and cannot be expressed as the union of two proper-open subsets  $V$  and  $W$  satisfying  $I^-(V) \subset V$

and  $I^-(W) \subset W$  respectively. Similarly using  $I^+$  one can define IF. The class of IPs can be divided into two classes: proper IPs (PIPs) which are of the form  $I^-(p)$  for  $p \in \mathcal{M}$ , and terminal IPs (TIPs) which are not formed by the history of any point in  $\mathcal{M}$ . We shall denote by  $\widehat{\mathcal{M}}$  the set of all IPs of the space  $(\mathcal{M}, g)$  and  $\mathcal{M}$  the set of all IFs of the space  $(\mathcal{M}, g)$ . We can now define a suitable topology on  $\widehat{\mathcal{M}}, \mathcal{M}$  to identify IFs and IPs [1] and one can form a space  $\mathcal{M}^* = \mathcal{M} \cup \Delta$  where  $\Delta$  is called the causal boundary.

The main subject of the present work is the  $b$ -boundary. This is a method developed by Schmidt which allows one to attach a boundary  $\partial\mathcal{M}$  called the  $b$ -boundary to any incomplete spacetime  $\mathcal{M}$  (or even to any manifold with a connection). The procedure consists on constructing a Riemannian metric for the frame bundle  $L(\mathcal{M})$  or the orthonormal bundle  $O(\mathcal{M})$  called the Schmidt metric. This metric is then used to generalise the idea of affine length to all curves. This generalisation is important because it helps to unify some elements of Riemannian geometry with Lorentzian geometry. For example, while only in the Riemannian case geodesic completeness implies that every curve is complete; the notion of  $b$ -completeness implies completeness of every curve in both signatures. The definition of a curve we are using here is a piecewise- $C^1$  curve. The  $b$ -boundary has also given some results that link the geometry of principal bundles with that of the base manifold [19] and with non-commutative geometry [20]. It would be tempting to establish a relationship between these three methods. However, to the best of our knowledge such relationship is not yet understood.

The structure of this paper is as follows. In Sec. II we give a general overview of the mathematical preliminaries needed. In section III we describe how the  $b$ -boundary is constructed following the procedure by Schmidt [15]. In Sec. IV we discuss the  $b$ -boundary construction for 1+1 conformally flat spacetimes and provide several examples. To our knowledge such calculations have not been done before and it will provide a better geometrical understanding of the  $b$ -boundary construction. In the last section, namely Sec. V, we discuss the topology of the completed manifold and the geometry of the orthonormal bundle. We also include two appendices. In the first appendix A we describe the Schmidt metric in the case of Riemannian surfaces. In the second appendix B where we give the classification of singularities using the  $b$ -boundary according to [21] [22].

## II. PRELIMINARIES

As a first step, let us present some of the required concepts of differential geometry. We present the basic concept of a metric on a manifold and the corresponding classification on Lorentzian and Riemannian manifolds. Then we move on to define fibre bundles and  $G$ -principal bundles. In the last part of this section we construct a set of familiar tensor and quasi-tensor quantities which are helpful when characterising manifolds with metrics. The manifolds we consider in this paper are paracompact,  $C^\infty$ , connected, and Hausdorff.

### A. Metric

A standard definition of *metric* is a symmetric and non-degenerate tensor  $g$  of type  $(0, 2)$  at each point  $p$  of a  $n$ -dimensional manifold  $\mathcal{M}$ . Given a basis  $\{\mathbf{E}^a\}$ , with  $a$  taking values  $0, 1, \dots, n-1$ , of the tangent space  $T_p$  at point  $p$ , the components of  $g$  are  $g_{ab} = g(\mathbf{E}^a, \mathbf{E}^b)$ . The signature of  $g$  at  $p$  is the number of positive eigenvalues minus the number of negatives ones. If  $g$  has signature  $n-2$  we call  $g$  a *Lorentzian metric*. For a Lorentzian metric on  $\mathcal{M}$ , the non-zero vectors at  $p$  can be divided into three classes: a vector  $\mathbf{X} \in T_p$  is said to be *timelike*, *null* or *spacelike* according to whether  $g(\mathbf{X}, \mathbf{X})$  is negative, zero or positive. On the other hand, if the signature is  $n$  then the metric is called *Riemannian metric*. A basis is *orthonormal* if  $g(\mathbf{E}^a, \mathbf{E}^b) = \eta_{ab}$  where  $\eta_{ab}$  is a diagonal matrix and  $|g(\mathbf{E}^a, \mathbf{E}^a)| = 1 \quad \forall a \in 0, 1, \dots, n-1$ .

If we are on a Riemannian manifold then the metric defines a *distance function*:

$$d(x, y) : (x, y) \in \mathcal{M} \times \mathcal{M} \rightarrow \inf_{\gamma(x, y)} \left\{ \int \|\dot{\gamma}(x, y)\| \right\} \in \mathbb{R},$$

where the infimum is taken over all piecewise  $C^1$  curves  $\gamma$  connecting point  $x$  to point  $y$ . Moreover, the distance function allows us to define a topology. A basis for this topology is given by the set  $\{B(x, r) : y \in \mathcal{M} | d(x, y) \leq r \quad \forall x \in \mathcal{M}\}$ . The topology naturally induces a notion of convergence: We say that the sequence  $\{x_n\}$  converges to  $y$  if for  $\epsilon > 0$  there is a value  $N \in \mathbb{N}$  such that for any  $n \geq N$ ,  $d(x_n, y) \leq \epsilon$ . A sequence satisfying these conditions is called a *Cauchy sequence*. If every Cauchy sequence converges we say that  $\mathcal{M}$  is *metrically complete*.

## B. Fibre Bundles and $G$ -Principal Bundles

A *Fibre bundle* with fibre  $\mathcal{F}$  is a manifold  $E$  with a surjective map  $\pi : E \rightarrow \mathcal{M}$  where there is a neighbourhood  $\mathcal{U}$  at each point  $p$  of  $\mathcal{M}$  such that  $\pi^{-1}(\mathcal{U})$  is isomorphic to  $\mathcal{U} \times \mathcal{F}$ , i.e., for each point  $p \in \mathcal{U}$  there is a diffeomorphism  $\phi_p$  of  $\pi^{-1}(p)$  onto  $\mathcal{F}$  such that the map  $\psi(\bar{p}) = (\pi(\bar{p}), \phi_{\pi(\bar{p})})$  is a diffeomorphism. We call  $\mathcal{M}$  the base space of the fibre bundle  $E$ .

A  $G$ -*principal bundle*  $P$  over a manifold  $\mathcal{M}$  is a fibre bundle with fibre a lie group  $G$  with a continuous right action  $R_g$  that acts freely:

$$(\bar{p}, g) \in P \times G \rightarrow R_g(\bar{p}) = \bar{p}g \in P$$

and satisfies that  $\mathcal{M}$  is the quotient space of  $P$  by the equivalence relation induced by  $G$  [23].

Let  $\mathcal{M}$  be a  $n$ -dimensional manifold. A *frame*  $\{\mathbf{E}^a\}_p$  at point  $p$  is an ordered basis of  $T_p$ . Let  $L(\mathcal{M})$  be the set of all frames  $\{\mathbf{E}^a\}$  at all points on  $\mathcal{M}$  with the projection  $\pi$  sending a frame at  $p$  to  $p$ . Then the *general linear group*  $GL(n, \mathbb{R})$  has a natural action on  $\{\mathbf{E}^a\}$ , i. e., given  $(\{\mathbf{E}^a\}, A_a^b)$  the action of  $A_a^b \in GL(n, \mathbb{R})$  on  $\{\mathbf{E}^a\}$  is  $\{\mathbf{E}^b = A_a^b \mathbf{E}^a\}$ . If  $\{x^a\}$  are coordinates on  $\mathcal{M}$  and we choose the frame  $\{\frac{\partial}{\partial x^a}\}$ , then it can be shown that the coordinates  $(x^a, \beta_b^a)$  are a local coordinate system of  $L\mathcal{M}$ , where  $\beta_b^a$  represent the  $ab$  element for the change of basis matrix  $\beta$  between  $\{\frac{\partial}{\partial x^a}\}$  and any other frame  $\{\mathbf{E}^b\}$ . In fact this choice makes  $L(\mathcal{M})$  a  $G$ -principal bundle called the *frame bundle*. Moreover, if we have a metric in  $\mathcal{M}$  and we restrict the frames to just orthonormal frames, we obtain another  $G$ -principal bundle called the *orthonormal frame bundle*  $O(\mathcal{M})$ . The associated Lie Group to  $O(\mathcal{M})$  is then the *orthonormal group*,  $SO(n, \mathbb{R})$  or  $SO^+(1, n, \mathbb{R})$ . This last definition is signature dependent.

Given a  $G$ -principal bundle  $P$  and a fibre bundle  $E$ , with fibre  $\mathcal{F}$  where  $P$  and  $E$  have the same base space  $\mathcal{M}$ , then  $E$  is called an *associated bundle* to  $P$  if the quotient space of  $P \times \mathcal{F}$  by the equivalence relation given by the action

$$(\bar{p}, f) \in P \times \mathcal{F} \rightarrow (\bar{p}g, fg^{-1}) \in P \times \mathcal{F}$$

and denoted as  $P \times_G \mathcal{F}$  is  $E$ . As an example of such construction let us mention that the tangent bundle  $T(\mathcal{M})$  of a  $n$ -dimensional manifold  $\mathcal{M}$  is isomorphic to an associated vector bundle of  $L(\mathcal{M})$  with fibre  $\mathbb{R}^n$ .

Every tangent space  $T_{\bar{p}}P$  of a  $G$ -principal bundle  $P$  has a subspace called the *vertical subspace*  $V_{\bar{p}}$ . This subspace is given by the kernel of the differential  $D\pi$  restricted at  $\bar{p}$ . Explicitly,

$$V_{\bar{p}} = \{\mathbf{X} \in T_{\bar{p}}P | D\pi_{\bar{p}}(\mathbf{X}) = 0 \in T_{\pi(\bar{p})}\mathcal{M}\}.$$

## C. Solder form and Connections

The *solder form* of a frame bundle  $L(\mathcal{M})$  is the map:

$$\theta : TL(\mathcal{M}) \rightarrow \mathbb{R}^n : (\bar{p}, Q) \rightarrow \pi_p(D\pi(\bar{p}, Q))$$

where  $\bar{p}$  is a point in  $L(\mathcal{M})$  and  $Q$  is an element of  $T_{\bar{p}}L\mathcal{M}$ . The solder form for the orthonormal bundle  $O(\mathcal{M})$  is defined similarly. Notice that  $V_{\bar{p}} \subset \ker(\theta)$ .

A *connection*  $\bar{\nabla}$  on a  $G$ -principal bundle is an assignment of a subspace  $H_{\bar{p}}$  called the *horizontal subspace* of  $T_{\bar{p}}(P)$  for all  $\bar{p}$  in  $P$  such that:

- $T_{\bar{p}}P = H_{\bar{p}} \oplus V_{\bar{p}}$ .
- For any  $\bar{p}, \bar{q} \in P$  there is a  $C^\infty$  curve  $\gamma$  such that  $T_{\gamma(t)}P = H_{\gamma(t)} \oplus V_{\gamma(t)}$ .
- $H_{\bar{p}g} = D_{\bar{p}}R_g(H_{\bar{p}})$  for every  $\bar{p} \in P$  and  $g \in G$ .

A *connection form*  $\varpi$  of a connection  $\bar{\nabla}$  in a  $G$ -principal bundle is a  $C^\infty$  map

$$\varpi : TP \rightarrow \mathfrak{g}$$

with the following properties:

- if  $\varpi(X) = 0$  then  $X \in H_{\bar{p}}$  for some  $\bar{p}$  in  $P$ ,
- for all  $g$  in  $G$  and all  $C^\infty$  maps  $X : P \rightarrow TP$ ,  $\varpi(DR_g(X)) = ad_*(g^{-1})\varpi(X)$ , and

- for all  $\vec{g} \in \mathfrak{g}$ ,  $\varpi(X_p^*) = \vec{g}$  where  $X_p^*$  is the tangent vector at  $t = 0$  of a curve given by  $\gamma(t) = R_{\exp t\vec{g}}(\bar{p})$ .

Let us remind the reader that connections and connection forms uniquely determine one another.

We shall now express the connection  $\varpi$  by a family of forms, where each family is defined in an open subset of the base manifold  $\mathcal{M}$ . Let  $\{\mathcal{U}_\alpha\}$  be an open covering of  $\mathcal{M}$  with a family of diffeomorphisms  $\psi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \rightarrow \mathcal{U}_\alpha \times G$  and the corresponding family of transition functions  $\psi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \rightarrow G$ . For each  $\alpha$ , let  $x_\alpha : \mathcal{U}_\alpha \rightarrow P$  be defined by  $x_\alpha =: \psi_\alpha^{-1}(x, e)$ ,  $x \in \mathcal{U}_\alpha$ , where  $e$  is the identity in  $G$ .

For each  $\alpha$ , define a connection form on  $\mathcal{U}_\alpha$  such that

$$\varpi_\alpha(v) := \varpi(Dx_\alpha(v))$$

for all  $v \in T\mathcal{U}_\alpha$ . Then  $\varpi_\alpha$  is a  $\mathfrak{g}$ -valued one-form. We also define on  $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ , a  $\mathfrak{g}$ -valued one-form by

$$\theta_\alpha(v) := \theta(D\psi_{\alpha\beta}(v)).$$

The family of one forms  $\{\varpi_\alpha, \theta_\alpha\}$  defines uniquely  $\varpi$  [23].

If we pick a particular chart  $\mathcal{U}_\alpha$  and a basis  $\{\mathbf{F}_b^a\}$  of  $\mathfrak{g}$  then it is possible to define  $n^3$  functions  $\Gamma_{bc}^a$ , on  $\mathcal{U}_\alpha$ . These functions are such that  $\varpi_\alpha(\cdot) = \sum_b (\Gamma_{bc}^a dx^b(\cdot)) \mathbf{F}_b^a$ , on  $\mathcal{U}_\alpha$ . The functions  $\Gamma_{bc}^a$  are called the *Christoffel symbols*.

In coordinates the connection form  $\varpi$  is written as  $\varpi = \sum_{a,b} \varpi_b^a \mathbf{F}_b^a$  where

$$\varpi_b^a = \sum_c \left( (\beta^{-1})_c^a dx^c + \sum_{d,e} (\beta^{-1})_c^a \Gamma_{de}^c \beta_b^e dx^d \right), \quad (1)$$

where  $(\beta^{-1})_c^a$  is the inverse of the matrix  $\beta_c^a$ .

The solder form  $\theta$  is then given by  $\theta = \sum_a \theta^a \mathbf{e}^a$  where

$$\theta^a = \sum_c (\beta^{-1})_c^a dx^c. \quad (2)$$

and  $\mathbf{e}^a$  is the natural basis of  $\mathbb{R}^n$

Just like in the case of the principle bundle, the connection forms  $\varpi_\alpha$  define a connection  $\nabla$ . The connection  $\nabla$  satisfies that:

- if  $\mathbf{T}$  is a  $C^r$  tensor of rank  $(r, s)$ , then  $\nabla \mathbf{T}$  is a  $C^{r-1}$  tensor field of type  $(r, s+1)$ ,
- $\nabla$  is linear,
- for arbitrary tensor fields  $\mathbf{T}$  and  $\mathbf{S}$ ,  $\nabla$  satisfies the Leibniz's property i.e.,  $\nabla(\mathbf{T} \otimes \mathbf{S}) = \nabla(\mathbf{T}) \otimes \mathbf{S} + \mathbf{T} \otimes \nabla(\mathbf{S})$ ,
- $\nabla f = df$  for any scalar function  $f$ .

If the  $G$ -principal bundle is  $L(\mathcal{M})$  we define a *covariant derivative* of a vector  $\mathbf{Y}$  along  $\mathbf{X}$  on the base manifold  $\mathcal{M}$  by the vector  $\nabla_{\mathbf{X}} \mathbf{Y} := \nabla \mathbf{Y}(\mathbf{X})$ . In general we define the covariant derivative of a tensor  $\mathbf{T}$  along  $\mathbf{X}$  to be the tensor  $\nabla_{\mathbf{X}} \mathbf{T} := \nabla \mathbf{T}(\mathbf{X})$  and we denote its components by  $T^{a_1 \dots a_r}_{b_1 \dots b_s; l}$ .

This derivative provides a notion of parallel propagation on the manifold  $\mathcal{M}$  along a curve  $\gamma$ . Let  $\gamma(t)$  be a  $C^1$  for an arbitrary parameter  $t$ . A vector  $\mathbf{V}$  satisfying

$$\nabla_{\dot{\gamma}} \mathbf{V} = 0$$

is said to be parallel along  $\gamma$ . If a curve  $\gamma$  with tangent vector  $\dot{\gamma}$  satisfies the equation

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \quad (3)$$

then  $\gamma$  is called a *geodesic*. The parameter  $t$  of  $\gamma(t)$  is called an *affine parameter* and the corresponding length to this parameter is the *affine length*. If we have a metric  $g$  on  $\mathcal{M}$  we say that the connection is *metric compatible* if  $\nabla_{\frac{\partial}{\partial x^c}} g = 0$  for any choice of basis  $\{\frac{\partial}{\partial x^c}\}$ . If in addition  $\nabla$  satisfies  $\nabla_{\mathbf{Y}} \mathbf{X} - \nabla_{\mathbf{X}} \mathbf{Y} = [\mathbf{X}, \mathbf{Y}]$  where  $[\mathbf{X}, \mathbf{Y}]$  is the Lie bracket of vector fields for any two vectors  $\mathbf{X}, \mathbf{Y}$  we call it the *Levi-Civita connection* of the metric  $g$ . This connection always exists and it is unique.

### D. Curvature

An important  $(3, 1)$ -tensor defined by the connection is the *Riemann tensor*, defined as

$$\mathbf{R}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) := \nabla_{\mathbf{X}}(\nabla_{\mathbf{Y}}\mathbf{Z}) - \nabla_{\mathbf{Y}}(\nabla_{\mathbf{X}}\mathbf{Z}) - \nabla_{[\mathbf{X}, \mathbf{Y}]\mathbf{Z}}$$

for any three arbitrary vectors  $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ . We can rewrite the above expression as

$$\sum_{b,c,d} R^a_{bcd} X^c Y^d Z^b := \sum_{b,c,d} (Z^a_{;dc} - Z^a_{;cd}) X^c Y^d,$$

where we have use the notation  $Z^a_{;bc}$  to denote the components of the  $\nabla\nabla\mathbf{Z}$  and  $R^a_{bcd}$  are the components of the Riemann tensor. The symmetries of the Riemann tensor implies that there are  $\frac{1}{12}n^2(n^2-1)$  algebraically independent components of  $R^d_{abc}$ , where  $n$  is again the dimension of  $\mathcal{M}$ . This tensor defines the curvature of the manifold. By contracting the curvature tensor, one can define a type  $(0, 2)$  tensor called the *Ricci tensor* with components

$$R_{bd} = \sum_a R^a_{bad}$$

which can represent  $\frac{1}{2}n(n+1)$  components of the curvature tensor. The *Ricci scalar*  $R$  is the contraction of the Ricci tensor:

$$R = \sum_a R^a_a = \sum_{a,b,d} R^a_{bad} g^{bd}$$

If  $n = 1$  then  $R^d_{abc} = 0$ ; if  $n = 2$  there is only one independent component of  $R^a_{bcd}$  which is essentially the function  $R$ . If  $n = 3$  all the information of the curvature is encoded in the Ricci tensor. As we will be interested in no higher dimension than  $n = 3$  the curvature will be determined completely using the Ricci tensor and the Ricci scalar.

### E. Cotton-York

In his doctoral thesis É. Cotton [24] showed that any 3-dimensional manifold is conformally flat if the covariant form  $C$  with components

$$C_{abc} = R_{ab;c} - R_{ac;b} + \frac{1}{4}((Rg_{ac})_{;b} - (Rg_{ab})_{;c})$$

vanishes.

Moreover, using the Hodge star operator one can convert the Cotton tensor into a trace free tensor called the Cotton-York tensor[25] with components:

$$C_a^b = \sum_{d,c} \left( R_{ca} - \frac{1}{4} R g_{ca} \right)_{;d} \eta^{dcb}. \quad (4)$$

The eigenvalues of the Cotton-Tensor allow us to classify manifolds with a metric according to [26]. For completeness we include such classification for the Riemannian case which is the relevant one in the present work:

- Class A: When the Cotton-York tensor has three different eigenvalues  $\lambda_i$  such that  $\lambda_1 \neq \lambda_2$  and  $\lambda_1 + \lambda_2 = -\lambda_3$ .
- Class B: When two of the eigenvalues of the Cotton-York tensor are repeated, i.e.,  $\lambda_1 = \lambda_2$  and  $\lambda_3 = -2\lambda_1$ .
- Class C: When all three eigenvalues are zero, i.e., the manifold is conformally flat.

## III. THE SCHMIDT METRIC

If one thinks of a singularity in classical Newtonian gravity, the statement that the gravitational field is singular at a certain location is unambiguous. As an example, take the gravitational potential of a spherical mass  $M$  in Cartesian coordinates

$$V(t, x, y, z) = \frac{GM}{\sqrt{x^2 + y^2 + z^2}},$$

where  $G$  is the gravitational constant, and the potential exhibits a singularity at the point  $x = y = z = 0$ , for any time  $t$  in  $\mathbb{R}$ . The location of the singularity is well defined because the coordinates have an intrinsic character which is independent of  $V$ .

However, in the case of GR the prescription given above can not work. This is due to the identification of the background spacetime with the gravitational field. Hence, only in the regions where the gravitational field is defined it is meaningful to talk about locations. Consider the spacetime with the line element

$$ds^2 = -\frac{1}{t^2}dt^2 + dx^2 + dy^2 + dz^2,$$

defined on the manifold  $\{(t, x, y, z) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}^3\}$ . If we say that there is a singularity at the point  $t = 0$ , we will be speaking too soon for two reasons. The first one is that  $t = 0$  is not part of the manifold. It makes no sense to talk about  $t = 0$  as a location where the field diverges. The second thing is that the lack of an intrinsic meaning of the coordinates in GR must be taken seriously. By making the coordinate transformation  $\eta = \log(t)$ , we obtain the line element

$$ds^2 = -d\eta^2 + dx^2 + dy^2 + dz^2,$$

on  $\mathbb{R}^4$  which is an isometric extension of the previously defined spacetime. This spacetime is, of course, Minkowski spacetime which is non-singular [1].

Another idea is trying to define a singularity in terms of invariant quantities such as invariant scalars. The reason for this is that if these quantities diverge then it matches our physical idea that objects must suffer stronger and stronger deformations as we encounter the singularity. These scalars are usually constructed from contractions of the Riemann tensor and its derivatives. Unfortunately, these scalars are not well-suited to define the complete geometry. Consider the metric

$$ds^2 = dudv + H_{ij}(u)x^i x^j du^2 - dx^i dx^i,$$

given in the coordinates  $(u, v, x^1, x^2)$  and where  $H(u)$  is  $C^1$ .

This spacetime is known as a *pp-wave* spacetime and it can be shown that every polynomial curvature-scalar vanishes, despite the fact that in general the spacetime is not flat [27].

A more troublesome feature of using scalars for defining singularities is that they are ‘too local’ in the sense that they are evaluated at given points. Therefore, if the point is removed, the scalar cannot be computed directly and we need an approximation procedure.

A precise mathematical way to approximate the “missing points” is to use convergent sequences of points on the manifold. In this case the formal statement is: “The sequence  $\{R(x_n)\}$  diverges while the sequence  $\{x_n\}$  converges to  $y$ ”, where  $R(x_n)$  is some scalar curvature invariant evaluated at  $x_n$  in  $\mathcal{M}$  and  $y$  is some point not necessarily in  $\mathcal{M}$ . In Riemannian geometry, the notion of distance allows us to define Cauchy sequences  $\{x_n\}$  and therefore a notion of convergence. Moreover, if every Cauchy sequence converges in  $\mathcal{M}$  then every geodesic can be extended indefinitely. This means we can take the domain of every geodesic to be  $\mathbb{R}$ . In this case, we say that  $\mathcal{M}$  is *geodesically complete*. In fact, the converse is also true: if  $\mathcal{M}$  is geodesically complete then  $\mathcal{M}$  is metrically complete, *i. e.*, every Cauchy sequence converges to a point in  $\mathcal{M}$  [28]. This allows us to use Cauchy sequences or sequences of points along geodesics as our sequences of points.

The Riemannian case is a useful example, but as soon as we move to Lorentzian geometry, which we take as the correct geometrical setting for GR, the previous discussion cannot be used as stated. The reason is that Lorentzian metrics do not have a distance function defined and therefore Cauchy sequences cannot be defined. Thus, one is restricted to the notion of geodesically complete manifolds in the Lorentzian case.

Moreover, the existence of three kinds of vectors available in any Lorentzian metric defines three nonequivalent notions of geodesic completeness —depending on the character of the tangent vector of the curve— spacelike completeness, null completeness and timelike completeness, which are, unfortunately, not equivalent. It is possible to construct spacetimes with the following characteristics [29–31]:

- timelike complete, spacelike and null incomplete,
- spacelike complete, timelike and null incomplete,
- null complete, timelike and spacelike incomplete,
- timelike and null complete, spacelike incomplete,
- spacelike and null complete, timelike incomplete, or

- timelike and spacelike complete, null incomplete.

Furthermore, there are examples of a geodesically null, timelike and spacelike complete spacetimes with an inextendible timelike curve of finite length [30, 31]. A particle following this trajectory will experience bounded acceleration and in a finite amount of proper time its spacetime location would stop being represented as a point in the manifold.

In order to overcome this, Schmidt provided an elegant way to generalise the idea of affine length to all curves, regardless of such curves being geodesic or not. This construction in the case of incomplete curves allows to attach to the spacetime  $\mathcal{M}$  a topological boundary  $\partial\mathcal{M}$  called the  $b$ -boundary. The procedure for constructing the Schmidt metric consists in building a Riemannian metric in the frame bundle  $L(\mathcal{M})$ . We use the solder form  $\theta$  on  $L(\mathcal{M})$  and the connection form  $\varpi$  on  $L(\mathcal{M})$  associated to the Levi-Civita connection  $\nabla$  on  $\mathcal{M}$  to do this. Explicitly, the Schmidt metric is given by

$$\bar{g}(\mathbf{X}, \mathbf{Y}) = \theta(\mathbf{X}) \cdot \theta(\mathbf{Y}) + \varpi(\mathbf{X}) \bullet \varpi(\mathbf{Y}), \quad (5)$$

where  $\mathbf{X}, \mathbf{Y} \in T_{\bar{p}}P$  and  $\cdot, \bullet$  are the inner products in  $\mathbb{R}^n$  and  $\mathfrak{g} \cong \mathbb{R}^{n^2}$  respectively. It can be shown that despite the freedom in the choice of the inner product, all the metrics constructed in this way are equivalent metrics.

Let  $\gamma : [a, b] \rightarrow \mathcal{M}$  be a piecewise- $C^1$  curve through  $p$  in  $\mathcal{M}$ . A curve  $\bar{\gamma} : [a, b] \rightarrow L(\mathcal{M})$  in  $L(\mathcal{M})$  is called the *lift of the curve*  $\gamma$  if it satisfies  $\pi(\bar{\gamma}) = \gamma$  and  $D\pi(\dot{\bar{\gamma}}) = \dot{\gamma}$ . The length of  $\bar{\gamma}$  with respect to the Schmidt metric, is

$$L_{\bar{\gamma}}(b) = \int_a^b \|\dot{\bar{\gamma}}(\eta)\|_{\bar{g}} d\eta,$$

which is called the *generalised affine-length* of  $\gamma$ . We can then use this to re-parametrise  $\gamma$  which generalises the notion of an affine parameter. In the case where  $\gamma$  is a geodesic parametrised by  $L_{\bar{\gamma}(t)}$ , it is parametrised with respect to an affine parameter. If every curve in a spacetime  $\mathcal{M}$  with finite generalised-affine-length has endpoints, we call this spacetime *b-complete*. If it is not *b-complete* we say that the spacetime is *b-incomplete*.

Notice that if there is a curve  $\gamma$  in  $\mathcal{M}$  that has finite affine-length and no endpoint then the lift curve  $\bar{\gamma}$  cannot have an endpoint. Otherwise, if  $\bar{p}$  is the endpoint of  $\bar{\gamma}$ ,  $\pi(\bar{p}) = p$  would be an endpoint of  $\gamma$  contradicting the incompleteness of  $\gamma$ . The previous remark shows that geodesic incompleteness implies *b-incompleteness*. The converse is not true as Geroch's example [30] shows a *b-incomplete* spacetime that is geodesically complete. Therefore *b-incompleteness* is a generalisation of geodesic incompleteness.

Now given an incomplete spacetime  $\mathcal{M}$ , using the Riemannian metric  $\bar{g}$  on  $L(\mathcal{M})$ , we can 'Cauchy complete'  $L(\mathcal{M})$ . Let us denote by  $\overline{L(\mathcal{M})}$  the Cauchy completion of  $L(\mathcal{M})$ .

We define the quotient space  $\overline{\mathcal{M}} = \overline{L(\mathcal{M})}/G^+$ , where  $G^+$  is the connected component of the identity of  $GL(n; \mathbb{R})$  under the equivalence of orbits, i.e.,  $(\bar{p}, g) \in \overline{L(\mathcal{M})} \sim (\bar{q}, g')$  if  $\bar{p} = \bar{q}$  and there is  $h \in GL(n; \mathbb{R})$  such that  $g = hg'$ . This quotient induces a topology in  $\overline{\mathcal{M}}$  by requiring that the map  $\pi : \overline{L(\mathcal{M})} \rightarrow \overline{\mathcal{M}}$  is continuous and therefore  $\overline{\mathcal{M}}$  is a topological space. However, it does not imply that  $\overline{\mathcal{M}}$  is a manifold. Finally we can characterise the  $b$ -boundary as the set  $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$ .

We repeat the same construction for subgroups of  $GL(n; \mathbb{R})$ . In particular, a common choice in the Lorentzian case is the subgroup of all Lorentz transformations preserving both orientation and direction of time, which is called the proper orthochronous Lorentz group and it is denoted by  $SO^+(1, n)$ . In a completely analogous way we can form the quotient  $\overline{\mathcal{M}} = \overline{O(\mathcal{M})}/SO^+(1, n; \mathbb{R})$  and define the  $b$ -boundary as the set  $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$ . The advantage of this construction is that  $O(\mathcal{M})$  is a manifold of dimension  $n + \frac{n(n-1)}{2}$  instead of the  $n + n^2$  dimensions of  $L(\mathcal{M})$ . Also, the construction can be carried in a manifold with a Riemannian metric, in that case  $\overline{\mathcal{M}}$  is homeomorphic to the Cauchy completion of  $\mathcal{M}$  [32]. This reinforces the conviction that the  $b$ -boundary is a natural way to attach boundaries to manifolds with connections.

#### IV. THE $b$ -BOUNDARY FOR 1+1 SPACETIMES

As it was mentioned in the introduction, the  $b$ -boundary is a mathematically elegant construction to attach a boundary to an incomplete spacetime. However, working directly with it is cumbersome because of the high dimensionality of the bundles involved. It is important to notice that while in the Riemannian case there are three conformally distinctly connected Riemann surfaces (the disc, the plane and the sphere) in the Lorentzian case there are infinitely many conformally distinctly simple-connected Lorentzian surfaces. This fact forces us to choose particular examples from all the available possibilities.

In this section, we locally construct the Schmidt metric for general 1+1 spacetimes and then we give several explicit examples. The first example is an explicit calculation of the geometric properties of the Schmidt metric ( $O(\mathcal{M})$ ,  $\bar{g}$ )



in the case of 2-dimensional Minkowski spacetime. Then we analyse Friedmann-Robertson-Walker (FRW) spacetimes under different scale factors and under different matter content. Finally, we examine De-Sitter and Anti-De Sitter spacetime. We briefly discuss the results of the curvature quantities in each case and the eigenvalues of the Cotton-York tensor.

**Notation:** We use overlines to denote the Riemannian geometric quantities that belongs to  $O(\mathcal{M})$  while the geometric quantities without any overline belong to the Lorentzian manifold  $\mathcal{M}$ .

### A. The Schmidt metric for 1+1 conformal spacetimes

Let  $\mathcal{M}$  be a 2-D manifold with a Lorentzian metric  $g$  and an orthonormal bundle  $O(\mathcal{M})$ . Then, we can find coordinates  $(v, w)$  which locally transform the line element of the metric  $g$  to the following form [33]:

$$ds^2 = \Omega^2(v, w)(-dv^2 + dw^2). \quad (6)$$

An orthonormal basis is then given by the vector fields

$$E_1 = \frac{1}{\Omega} \frac{\partial}{\partial v} \quad \text{and} \quad (7)$$

$$E_2 = \frac{1}{\Omega} \frac{\partial}{\partial w}. \quad (8)$$

The orthonormal basis prescribed above is not unique. Any other orthonormal basis is of the form

$$\tilde{E}_1 = \cosh \chi \frac{1}{\Omega} \frac{\partial}{\partial v} + \sinh \chi \frac{1}{\Omega} \frac{\partial}{\partial w} \quad (9)$$

$$\tilde{E}_2 = \cosh \chi \frac{1}{\Omega} \frac{\partial}{\partial w} + \sinh \chi \frac{1}{\Omega} \frac{\partial}{\partial v} \quad (10)$$

for some  $\chi \in \mathbb{R}$ .

Let us notice that the coefficients of such a basis with respect to  $\frac{\partial}{\partial v}, \frac{\partial}{\partial w}$  define a unique non-singular matrix  $\beta$  with inverse  $\beta^{-1}$ :

$$\beta = \frac{1}{\Omega} \begin{pmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{pmatrix}, \quad \text{and} \quad \beta^{-1} = \Omega \begin{pmatrix} \cosh \chi & -\sinh \chi \\ -\sinh \chi & \cosh \chi \end{pmatrix}. \quad (11)$$

These matrices are important in the sense that they are useful to define local coordinates on  $O(\mathcal{M})$  as follows:

$$\left\{ v, w, \frac{1}{\Omega} \left( \cosh \chi \frac{\partial}{\partial v} + \sinh \chi \frac{\partial}{\partial w} \right), \frac{1}{\Omega} \left( \cosh \chi \frac{\partial}{\partial w} + \sinh \chi \frac{\partial}{\partial v} \right) \mid (v, w) \in \mathcal{M}, \chi \in \mathbb{R} \right\}. \quad (12)$$

As stated in section III, the Schmidt metric  $\bar{g}$  for any  $\mathbf{X}, \mathbf{Y} \in TO(\mathcal{M})$  on  $O(\mathcal{M})$  is given by

$$\bar{g}(\mathbf{X}, \mathbf{Y}) : \varpi(\mathbf{X}) \cdot \varpi(\mathbf{Y}) + \theta(\mathbf{X}) \cdot \theta(\mathbf{Y}) \quad (13)$$

where  $\varpi$  is the connection form on  $O(\mathcal{M})$  and  $\theta$  the solder form.

Now let us consider a curve  $\bar{\gamma}(s)$  in  $O(\mathcal{M})$  given by  $\bar{\gamma} : s \in [a, b] \mapsto (v(s), w(s), \beta_c^a(s))$  and evaluate  $\theta(\dot{\bar{\gamma}})$  and  $\varpi(\dot{\bar{\gamma}})$ . Explicitly we have

$$\theta(\dot{\bar{\gamma}}) = \Omega \begin{pmatrix} \dot{v} \cosh \chi - \dot{w} \sinh \chi \\ -\dot{v} \sinh \chi + \dot{w} \cosh \chi \end{pmatrix}, \quad (14)$$

and

$$\varpi(\dot{\bar{\gamma}}) = \begin{pmatrix} 0 & \dot{\chi} + \frac{1}{\Omega}((\partial_v \Omega)\dot{w} + (\partial_w \Omega)\dot{v}) \\ \dot{\chi} + \frac{1}{\Omega}((\partial_v \Omega)\dot{w} + (\partial_w \Omega)\dot{v}) & 0 \end{pmatrix}, \quad (15)$$

where we have used 1 and 2. Then, the line element for the Schmidt metric is:

$$ds^2 = \Omega^2(v, w)(\cosh(2\chi)(dv^2 + dw^2) - 2\sinh(2\chi)dv dw) + \left( d\chi + \frac{1}{\Omega} \left( \frac{\partial \Omega}{\partial v} dw + \frac{\partial \Omega}{\partial w} dv \right) \right)^2. \quad (16)$$

Using computer algebra we obtained analytical expressions for the curvature quantities for Eq. (16). We avoid quoting long tensorial expressions for the curvature tensors and give only the result for the Ricci scalar of (16) in terms of  $\Omega$  and its derivatives. Explicitly  $\bar{R}$  is given by

$$\bar{R} = -\frac{1}{2\Omega^8} ((\Omega_{ww} - \Omega_{vv})\Omega - (\Omega_w^2 - \Omega_v^2))^2 - 2. \quad (17)$$

Taking into account that

$$R = -\frac{2}{\Omega^4} ((\Omega_{ww} - \Omega_{vv})\Omega - (\Omega_w^2 - \Omega_v^2)) \quad (18)$$

This means that Eq. (17) becomes

$$\bar{R} = -\frac{1}{8}R^2 - 2 \quad (19)$$

The result of Eq. (19) straightforwardly gives an upper bound for the Ricci scalar for any Schmidt metric. This bound is  $\bar{R} = -2$  which is saturated in the Minkowski limit.

### B. The Schmidt metric of Minkowski spacetime

We now develop here in detail the geometric properties of the Schmidt metric for the case of Minkowski spacetime which is the simplest model of a Lorentzian manifold. The spacetime can be characterised by the condition  $R_{bcd}^a = 0$ , i.e., Minkowski spacetime is the only flat Lorentzian manifold. This spacetime is also the one of Special Relativity in which —although being a relativistic theory— the curvature effects produced by matter or gravity are not included. For the  $1+1$  case, the line element takes the form  $ds^2 = -dt^2 + dx^2$  where  $(t, x) \in \mathbb{R}^2$ . Using (16) we can write the Schmidt metric in the form

$$ds^2 = (dt^2 + dx^2) \cosh(2\chi) - 2dt dx \sinh(2\chi) + d\chi^2. \quad (20)$$

Now let us consider the change of coordinates:  $t = u + \tilde{v}$ ,  $x = u - \tilde{v}$  and write:

$$ds^2 = 2(\cosh 2\chi + \sinh 2\chi)du^2 + 2(\cosh 2\chi - \sinh 2\chi)d\tilde{v}^2 + d\chi^2 \quad (21)$$

or in an equivalent manner

$$ds^2 = 2e^{2\chi}du^2 + 2e^{-2\chi}d\tilde{v}^2 + d\chi^2. \quad (22)$$

We explicitly calculate  $\bar{R}_{ab}$  and get

$$\bar{R}_{\chi\chi} = -2, \quad (23)$$

and all other components are zero. The Ricci scalar is then

$$\bar{R} = -2. \quad (24)$$

Hence, the geometry in the bundle is not flat even if Minkowski spacetime is flat. The eigenvalues of the Cotton-York tensor computed according to Eq. (4) are

$$\lambda_1 = -2, \quad \lambda_2 = 0, \quad \text{and} \quad \lambda_3 = 2 = -(\lambda_1 + \lambda_2).$$

According to the classification quoted in Sec. II E this spacetime falls within Class A

### C. The Schmidt metric of Friedmann-Robertson-Walker (FRW) spacetime

The FRW spacetime corresponds to the model of a non-stationary, spatially homogeneous and isotropic universe. The observation of the cosmic microwave background [34] radiation and the red shift of type Ia supernovae [35] correspond to the strongest evidence that sustains a model of a non-stationary, homogeneous and isotropic universe. For a universe with topology  $\mathbb{R} \times \mathbb{R}^3$  the FRW metric is given by the line element

$$ds^2 = -dt^2 + a^2(t)dx^2 + a^2(t)dy^2 + a^2(t)dz^2, \quad (25)$$

where  $x, y, z$  are Cartesian coordinates on  $\mathbb{R}^3$ ,  $t \in \mathbb{R}^+$  is the temporal coordinate and  $a(t)$  is called the *scale factor*. The physical requirements of positive matter density, non-negative pressure and the observed recession of galaxies, leads to the behaviour  $a(t) \rightarrow 0$  as  $t \rightarrow 0$ . For a radiation-dominated universe the evolution of the scale factor is  $a(t) \propto t^{1/2}$ . For a matter-dominated universe the evolution of the scale factor is  $a(t) \propto t^{2/3}$ . For a dark-energy-dominated universe, the evolution of the scale factor is  $a(t) \propto \exp(Ht)$ . The coefficient  $H$  in the exponential is called the Hubble constant and it is given by  $H = \sqrt{\Lambda/3}$  where  $\Lambda$  is the cosmological constant. In the early universe after inflation the universe was dominated by radiation. Motivated by this, let us consider that the scale factor takes the form  $a(t) = t^\rho$  ( $\rho > 0$ ) as  $t \rightarrow 0$ . By rescaling the time coordinate we can put the metric in the form

$$ds^2 = -a^2(\eta)d\eta^2 + a^2(\eta)dx^2 + a^2(\eta)dy^2 + a^2(\eta)dz^2, \quad (26)$$

with  $a^2(\eta) = \eta^q$  ( $q > 0$ ).

For simplicity, let us consider the case of the 1 + 1 FRW cosmological model which can be obtained from the 4-dimensional one by collapsing two spatial coordinates. This is equivalent to considering the injection map

$$h : (\eta, x) \rightarrow (\eta, x, y_0, z_0) : \mathcal{M} \rightarrow \mathcal{N} = \mathcal{M} \times \Sigma, \quad (27)$$

where  $\Sigma$  is a suitable two dimensional manifold. This way Eq. (26) reduces to

$$ds^2 = \eta^q(-d\eta^2 + dx^2), \quad (28)$$

for any value of  $q > 0$ . The Ricci scalar corresponding to the spacetime described by (28) is

$$R = -q\eta^{-2-q}. \quad (29)$$

From Eq. (16) the Schmidt metric in  $O(\mathcal{M})$ , for our case study, takes the form

$$ds^2 = \eta^q (\cosh(2\chi)(d\eta^2 + dx^2) - 2 \sinh(2\chi)d\eta dx) + \left(d\chi + \frac{q}{\eta}dx\right)^2. \quad (30)$$

Using computer algebra we calculated the Ricci tensor for the line element (30). In components it is given by

$$\begin{aligned} \bar{R}_{\eta\eta} &= \frac{1}{8}q\eta^{-4-q}(4\eta^{2+q} - q \cosh(2\chi)), \\ \bar{R}_{xx} &= \frac{1}{32}q\eta^{-2(3+q)}(q^3 - 16\eta^{4+2q} - 16q\eta^{4+2q} - 4q(5+2q)\eta^{2+q} \cosh(2\chi)), \\ \bar{R}_{\chi\chi} &= -2 + \frac{1}{8}q^2\eta^{-2(2+q)}, \\ \bar{R}_{\eta x} &= \frac{1}{8}q^2(3+q)\eta^{-4-q} \sinh(2\chi), \\ \bar{R}_{\eta\chi} &= \frac{1}{4}q(2+q)\eta^{-3-q} \sinh(2\chi), \\ \bar{R}_{x\chi} &= \frac{1}{16}q\eta^{-5-2q}(q^2 - 16\eta^{4+2q} - 4(2+q)\eta^{2+q} \cosh(2\chi)). \end{aligned}$$

And the Ricci scalar is

$$\bar{R} = -2 - \frac{1}{8}q^2\eta^{-2(2+q)}, \quad (31)$$

which can equivalently be obtained from Eq. (19).

The eigenvalues of the Cotton-York tensor are checked numerically for the range of values in  $\eta \in (1, 10)$ ,  $q \in (1, 10)$  and  $\chi \in (0, 2\pi)$ . As an example we give the values when  $\eta = 1$ ,  $q = 2$  and  $\chi = \pi$

$$\lambda_1 = -3751.32, \quad \lambda_2 = 2.07, \quad \text{and} \quad \lambda_3 = 3749.25.$$

At each combination of  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  we check numerically that  $\lambda_1 + \lambda_2 = -\lambda_3$ . According to the classification of Sec. II E this spacetime once again falls within Class A

To get a visual notion of the behaviour of the Ricci scalar with respect of the parameter  $q$  we plot  $\bar{R}$  as a function of  $\eta$  for a small set of values of the parameter  $q > 0$ . This is shown in Fig. 1 where we appreciate that when  $q \rightarrow 0$  we recover the expected value for Minkowski spacetime:  $\bar{R} \rightarrow -2$ . We also bring attention to Fig. 2 which shows the behaviour of the curvature in a semi-log scale. Finally, we show the behaviour of the geometry in Fig. 3 where we have chosen some values with  $q < 0$  although these values are unphysical by the reasons above mentioned.

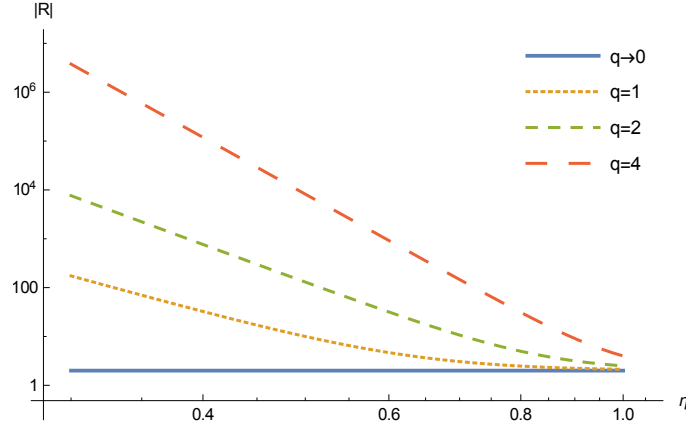


Figure 1: The Ricci scalar for the Schmidt metric of FRW as given by Eq. (31) in a log-log scale against the conformal time  $\eta$ . For different cases of  $q > 0$  the Ricci approaches asymptotically to a limiting case as the parameter  $\eta$  increases monotonically. Such limit occurs when  $\eta^q \rightarrow 1$  as  $q \rightarrow 0$  which has the constant Minkowski value  $\bar{R} = -2$  shown as the blue (solid) line.

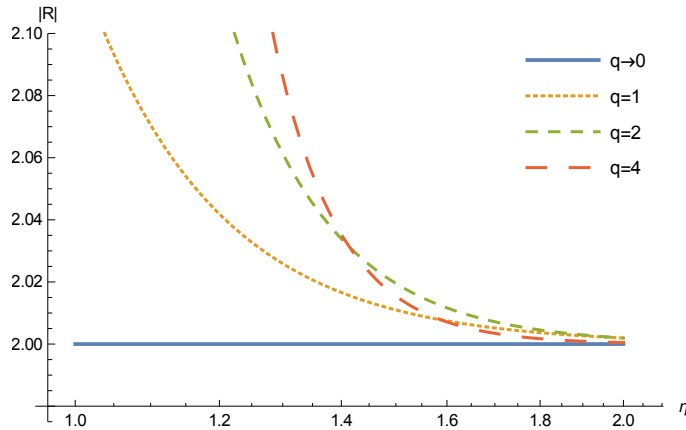


Figure 2: Ricci for the Schmidt metric of FRW as given by Eq. (31) in a semilog scale. The Minkowski limiting case is given by the blue (solid) line at  $\bar{R} = -2$  as before. As  $\eta$  increases the convergence to the Minkowski limit is faster for larger values of  $q$ .

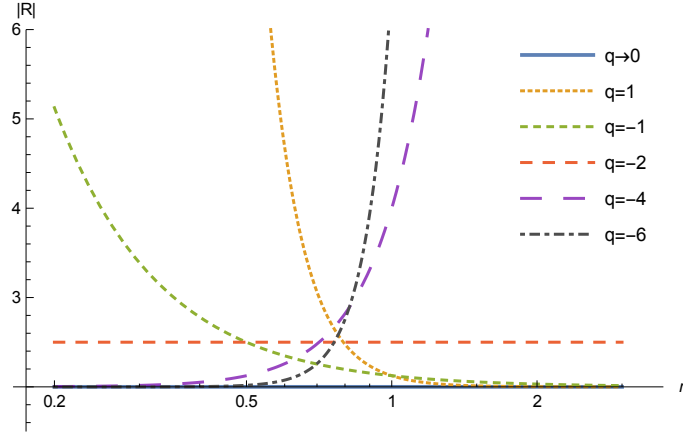


Figure 3: The Ricci for the Schmidt metric of FRW as given by Eq. (30) in a semilog scale. The physically relevant cases correspond to  $q \geq 0$ . The Minkowski limiting case is given by the blue (solid) line at  $\bar{R} = -2$  as before. When  $q < 0$  we distinguish two regimes: for values of  $0 > q > -2$  such as  $q = -1$  (yellow-dashed curve) the behaviour of the Ricci scalar follows the same rules as the positive cases; for  $q = -2$  (red-dashed curve) we have a separatrix of the two regimes, which gives a constant curvature  $\bar{R} = -2.5$ ; for values of  $q < -2$  the Ricci scalar start in the limit case and rapidly begin to grow until they diverge, this divergence is faster as  $|q|$  increases.

#### Ricci Scalar for radiation-dominated universe

Let us now consider an early era of the universe, where the dynamics was dominated by radiation. This case study is modelled as a particular version of FRW spacetime which can be recovered by fixing the conformal time  $\eta = 2t^{1/2}$ , the (conformal) scale factor takes the form  $a^2(\eta) = \frac{\eta^2}{4}$ . Then Eq. (26) turns into

$$ds^2 = \frac{\eta^2}{4}(-d\eta^2 + dx^2), \quad (32)$$

with Ricci scalar

$$R = -\frac{8}{\eta^4}. \quad (33)$$

Notice that the line element (32) is similar enough to (30). It is then straightforward to obtain the Schmidt metric of  $O(\mathcal{M})$  from Eq. (16):

$$ds^2 = \frac{\eta^2}{4}(\cosh(2\chi)(d\eta^2 + dx^2) - 2\sinh(2\chi)d\eta dx) + (d\chi + \frac{1}{\eta}dx)^2.$$

As before we quote here the components of the Ricci Tensor obtained using computer algebra:

$$\begin{aligned} \bar{R}_{\eta\eta} &= \frac{\eta^4 - 2\cosh(2\chi)}{\eta^6}, & \bar{R}_{xx} &= \frac{8 - 3\eta^8 - 18\eta^4\cosh(2\chi)}{\eta^{10}}, & \bar{R}_{\chi\chi} &= -2 + \frac{8}{\eta^8}, & \bar{R}_{\eta\chi} &= \frac{8\sinh(2\chi)}{\eta^5}, \\ \bar{R}_{\eta x} &= \frac{20\cosh(\chi)\sinh(\chi)}{\eta^6}, & \bar{R}_{x\chi} &= -\frac{2(\eta^8 - 4 + 4\eta^4\cosh(2\chi))}{\eta^9}. \end{aligned}$$

And the corresponding Ricci Scalar turns out to be

$$\bar{R} = -2 - \frac{8}{\eta^8}. \quad (34)$$

The eigenvalues of the Cotton-York tensor are once again checked numerically for the range of values in  $\eta \in (1, 10)$  and  $\xi \in (0, 2\pi)$ . As an example we give the values when  $\eta = 2$  and  $\xi = \pi$

$$\lambda_1 = -234.923, \quad \lambda_2 = 0.609, \quad \text{and} \quad \lambda_3 = 234.314.$$

At each combination of  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  we check numerically that  $\lambda_1 + \lambda_2 = -\lambda_3$ . According to the classification of Sec. II E this spacetime once again falls within Class A

*Ricci Scalar for matter-dominated universe*

When the Universe is in a matter-dominated era, the spacetime is such that the conformal time (of the FRW model) is taken as  $\eta = 3t^{1/3}$  giving  $a^2(\eta) = \frac{\eta^4}{81}$ . Therefore the metric of this cosmological model is

$$ds^2 = \frac{\eta^4}{81}(-d\eta^2 + dx^2),$$

with Ricci scalar

$$R = -\frac{324}{\eta^6}. \quad (35)$$

Again Eq. (16) gives the corresponding Schmidt metric:

$$ds^2 = \frac{\eta^4}{81}(\cosh(2\chi)(d\eta^2 + dx^2) - 2\sinh(2\chi)d\eta dx) + (d\chi + \frac{2}{\eta}dx)^2.$$

For this case study the components of the Ricci Tensor are

$$\begin{aligned} \bar{R}_{\eta\eta} &= \frac{2(\eta^6 - 81 \cosh(2\chi))}{\eta^8}, & \bar{R}_{xx} &= -\frac{2(-26244 + 5\eta^{12} + 1053\eta^6 \cosh(2\chi))}{\eta^{14}}, \\ \bar{R}_{\chi\chi} &= -2 + \frac{13122}{\eta^{12}}, & \bar{R}_{\eta\chi} &= \frac{486 \sinh(2\chi)}{\eta^7}, & \bar{R}_{\eta x} &= \frac{2268 \cosh(\chi) \sinh(\chi)}{\eta^8}, \\ \bar{R}_{x\chi} &= \frac{26244 - 4\eta^{12} - 486\eta^6 \cosh(2\chi)}{\eta^{13}}. \end{aligned}$$

And the corresponding trace equals to

$$\bar{R} = -2 - \frac{13122}{\eta^{12}}. \quad (36)$$

The eigenvalues of the Cotton-York tensor are once again checked numerically for the range of values in  $\eta \in (1, 10)$  and  $\xi \in (0, 2\pi)$ . As an example we give the values when  $\eta = 1.5$  and  $\xi = \pi$

$$\lambda_1 = -55893.3, \quad \lambda_2 = -87.5, \quad \text{and} \quad \lambda_3 = 55980.8.$$

At each combination of  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  we check numerically that  $\lambda_1 + \lambda_2 = -\lambda_3$ . According to the classification of Sec. II E this spacetime once again falls within Class A

*Ricci Scalar for a dark-energy-dominated universe*

For the dark-energy-dominated universe the scale factor is taken as  $a(t) = \exp(Ht)$ . The transformation to conformal time is given by  $\eta = \frac{-1}{H}e^{-Ht}$  while the scale factor is  $a^2(\eta) = \frac{1}{H^2\eta^2}$ . Starting from (26) and considering a 1 + 1 FRW cosmological model, the line element takes the form

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + dx^2),$$

with Ricci scalar

$$R = 2H^2. \quad (37)$$

Let us now write the Schmidt metric from (16). It explicitly takes the form

$$ds^2 = \frac{1}{H^2\eta^2}(\cosh(2\chi)(d\eta^2 + dx^2) - 2\sinh(2\chi)d\eta dx) + (d\chi - \frac{1}{\eta}dx)^2, \quad (38)$$

We compute Ricci Tensor, once again with the help of computer algebra, and obtain the non vanishing components

$$\begin{aligned}\bar{R}_{\eta\eta} &= -\frac{2 + H^2 \cosh(2\chi)}{2\eta^2}, & \bar{R}_{xx} &= \frac{-2 + H^4 - H^2 \cosh(2\chi)}{2\eta^2}, \\ \bar{R}_{\chi\chi} &= -2 + \frac{H^4}{2}, & \bar{R}_{\eta x} &= -\frac{-4 + H^4}{2\eta},\end{aligned}$$

with scalar curvature

$$\bar{R} = -2 - \frac{H^4}{2}. \quad (39)$$

For a dark-energy dominated FRW we get simple analytical expressions for the eigenvalues of the Cotton-York tensor. These are

$$\lambda_1 = -\frac{H^4}{\eta^2}, \quad \lambda_2 = \frac{4 - H^4 + H^6}{2H^2\eta^2}, \quad \text{and} \quad \lambda_3 = \frac{-4 + H^4 + H^6}{2H^2\eta^2} = -(\lambda_1 + \lambda_2).$$

With these eigenvalues the model defined by Eq. (38) falls within Class A.

In Fig. 4 we show the behaviour for the Ricci scalar of Eqs. (34),(36) and (39) of the three models mentioned above.

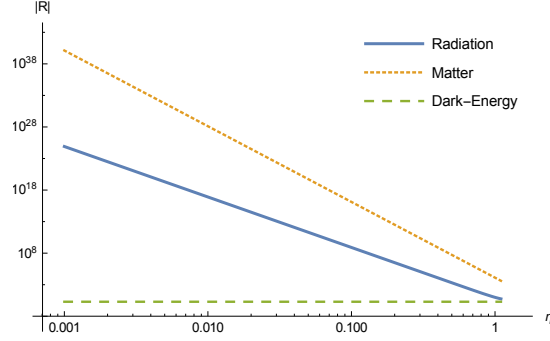


Figure 4: The Ricci scalar for the Schmidt metric of FRW for the cases when there's matter domain (yellow-dashed line), radiation domain (blue line) or dark-energy domain (green-dashed line) in a log-log scale. For the dark-energy case the curvature is always constant. For the radiation and matter cases the curvature diverges as  $\eta \rightarrow 0$ . The curvature in the matter and radiation domain approaches asymptotically to  $|\bar{R}| = 2$  as  $\eta$  increases, in other words, while the conformal time moves forward the Ricci scalar slowly goes to the Minkowski limit.

Fig. 5 shows that in the case of dark-energy domain the value of  $\bar{R}$  is constant for all values of time and it does not deviate far from the Minkowski value. This is to be expected as the value of  $H$  is close zero. In order to compare the remaining two cases we show the plot change our range of values for  $\bar{R}$ . In Fig. 5 we can see that  $\bar{R} \rightarrow -\infty$  as  $\eta \rightarrow 0$  when the model corresponds to a matter domain or a radiation domain. The radiation domain divergence for  $\bar{R}$  is slower.

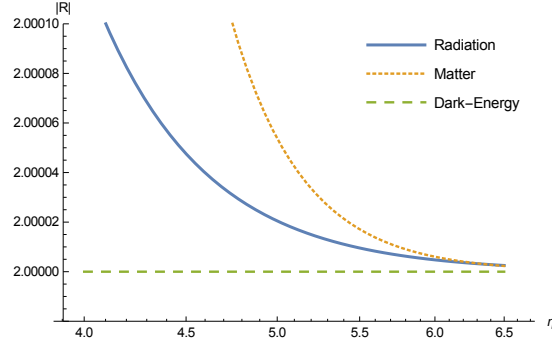


Figure 5: The Ricci scalar for the Schmidt metric of FRW for the cases when there's matter domain (yellow-dashed line), radiation domain (blue line) or dark-energy domain (green-dashed line) in a semilog scale. As  $\eta$  increases, the Ricci for the radiation and matter domains asymptotically approach the Minkowski limit, while the dark-energy case the curvature stay constant and is slightly smaller (of the order of  $\sim 10^{-13}$ ) than the Minkowski limit.

#### D. The Schmidt metric of De Sitter and Anti-De Sitter spacetimes

Let us now consider the De Sitter and Anti-De Sitter models and study the behaviour of the corresponding curvature scalars. First, consider the De Sitter case. The two dimensional De Sitter spacetime for closed spatial sections [36] can be defined with the line element

$$ds^2 = -d\tau^2 + \alpha^{-2} \cosh^2(\alpha\tau) d\omega^2.$$

To obtain the conformal form we make the change  $\tan(\eta/2) = \tanh(\alpha\tau/2)$ , which leads to

$$ds^2 = \frac{1}{\alpha^2 \cos^2(\eta)} (-d\eta^2 + d\omega^2). \quad (40)$$

In these coordinates  $(\eta, \omega)$  De Sitter space is conformal to the static Einstein universe [1]. The Ricci scalar for (40) is then

$$R = 2\alpha^2. \quad (41)$$

Using Eq. (16) we get

$$ds^2 = \frac{1}{\alpha^2 \cos^2(\eta)} (\cosh(2\chi)(d\eta^2 + d\omega^2) - 2 \sinh(2\chi) d\eta d\omega) + (d\chi - \tan(\eta) d\omega)^2. \quad (42)$$

The Ricci tensor is computed by taking  $\Omega = \frac{1}{\alpha \cos(\eta)}$ . The non-vanishing components are:

$$\begin{aligned} \bar{R}_{\eta\eta} &= -(1 + \frac{\alpha^2}{2} \cosh(2\chi)) \sec^2(\eta), & \bar{R}_{\omega\omega} &= \frac{\alpha^4}{4} (1 + (-1 + 4\alpha^{-4}) \cos(2\eta) - 2\alpha^{-2} \cosh(2\chi)) \sec^2(\eta), \\ \bar{R}_{\chi\chi} &= -2 + \frac{\alpha^4}{2}, & \bar{R}_{\eta\omega} &= \alpha^2 \cosh(\chi) \sec^2(\eta) \sinh(\chi), & \bar{R}_{\omega\chi} &= (-2 + \frac{\alpha^4}{2}) \tan(\eta). \end{aligned}$$

Thus the Ricci scalar is

$$\bar{R} = -2 - \frac{\alpha^4}{2}.$$

Notice that in the limit as  $\alpha \rightarrow 0$  we recover the Minkowski limit once again.

The eigenvalues of the Cotton-York tensor are

$$\lambda_1 = -\alpha^4 \sec^2 \eta, \quad \lambda_2 = \frac{(4 - \alpha^4 + \alpha^6) \sec^2 \eta}{2\alpha^2}, \quad \text{and} \quad \lambda_3 = \frac{(-4 + \alpha^4 + \alpha^6) \sec^2 \eta}{2\alpha^2} = -(\lambda_1 + \lambda_2). \quad (43)$$



Now let us look at the Anti-De Sitter spacetime. The two dimensional Anti-De Sitter metric has the line element

$$ds^2 = \frac{1}{\alpha^2 y^2} (-dt^2 + dy^2)$$

with  $y > 0$ . The Ricci scalar is

$$R = -2\alpha^2. \quad (44)$$

We identify  $\Omega = \frac{1}{\alpha y}$  and using Eq. (16) we obtain

$$ds^2 = \frac{1}{\alpha^2 y^2} (\cosh(2\chi)(dt^2 + dy^2) - 2 \sinh(2\chi) dt dy) + (d\chi - \frac{1}{y} dt)^2.$$

Where the non-vanishing components of the Ricci tensor are

$$\begin{aligned} \bar{R}_{tt} &= \frac{-2 + \alpha^4 - \alpha^2 \cosh(2\chi)}{2y^2}, & \bar{R}_{yy} &= -\frac{2 + \alpha^2 \cosh(2\chi)}{2y^2}, & \bar{R}_{\chi\chi} &= -2 + \frac{\alpha^4}{2}, \\ \bar{R}_{ty} &= \frac{\alpha^2 \cosh(\chi) \sinh(\chi)}{y^2}, & \bar{R}_{t\chi} &= -\frac{-4 + \alpha^4}{2y}, \end{aligned}$$

and its trace is given by

$$\bar{R} = -2 - \frac{\alpha^4}{2}.$$

And the eigenvalues of the Cotton-York tensor are

$$\lambda_1 = -\frac{2\alpha^6 y + 6\alpha^4 y(4+y) \cosh(2\chi) + \sqrt{2y^2(32 + 16\alpha^4(11+3y) + \alpha^8(98 + 60y + 9y^2) + 9\alpha^8(4+y)^2 \cosh(4\chi))}}{4\alpha^2 y^3}, \quad (45)$$

$$\lambda_2 = \frac{\alpha^2(\alpha^2 + 3(4+y) \cosh(2\chi))}{y^2}, \quad \text{and} \quad (46)$$

$$\lambda_3 = \frac{-2\alpha^6 y - 6\alpha^4 y(4+y) \cosh(2\chi) + \sqrt{2y^2(32 + 16\alpha^4(11+3y) + \alpha^8(98 + 60y + 9y^2) + 9\alpha^8(4+y)^2 \cosh(4\chi))}}{4\alpha^2 y^3}. \quad (47)$$

where again  $\lambda_3 = -(\lambda_1 + \lambda_2)$ . It follows that both spacetimes as defined by the Schmidt metric of De Sitter and Anti-De Sitter fall within Class A.

## V. DISCUSSION

In our exposition about the  $b$ -boundary, we obtained Eq. (16) which is the line element of the Schmidt metric for all  $1+1$  Lorentzian manifolds  $\mathcal{M}$ . This line element determines, via the curvature, all the local isometric invariants. If  $\partial\mathcal{M} = \emptyset$ , then the 3-manifold corresponds to the orthonormal bundle where the fibres of the bundle are  $SO^+(1,1) \cong \mathbb{R}$ . Therefore,  $O(\mathcal{M})$  is not compact. If  $\partial\mathcal{M} \neq \emptyset$ , then the 3-manifold is not necessarily a  $G$ -bundle (the group may not act freely or transitively). This is, for example, the case when  $\mathcal{M}$  is the Friedmann-Robertson-Walker spacetime. It also affects the topology of  $\bar{\mathcal{M}}$ , which in the Friedmann-Robertson-Walker case is no longer Hausdorff [32]. Then  $\overline{O(\mathcal{M})}$  is not a  $G$ -bundle as the fibre over the singularity is a point instead of a copy of  $SO(1,1)^+(\mathcal{M})$ . Nevertheless,  $\overline{O(\mathcal{M})}$  is still a 3-dimensional Riemannian non-compact manifold. In general, the topology of  $\bar{\mathcal{M}}$  is poorly understood. It is known that in four dimensions the Schwarzschild and the Friedmann-Robertson-Walker  $b$ -completions result in non-Hausdorff spaces[32]. But it is still a conjecture that in the latter  $\partial\mathcal{M}$  is a line while in the former  $\partial\mathcal{M}$  is a point [32, 37]. In fact there are general geometric conditions on the curvature to guarantee that the fibres above a boundary point are degenerate [38].

In this paper we have explore the geometry of the 3-manifold  $\overline{O(\mathcal{M})}$ . We have shown that for a wide class of spacetime metrics the eigenvalues of the Cotton-York tensor correspond to Class A (three distinct eigenvalues). This is a strong evidence to assume that the general Schmidt metric (16) falls also within Class A. Moreover we have found that the Ricci scalar of the Schmidt metric is bounded by  $-2$  and that Minkowski spacetime saturates the bound. Finally we notice that the Ricci scalars on  $O(\mathcal{M})$  for Anti-De Sitter and De Sitter space time are equal.

## VI. ACKNOWLEDGEMENTS

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### Appendix A: The Riemannian case

As it was mentioned in Sec. IV, in the Riemannian case there are three conformally distinctly connected Riemann surfaces (the disc, the plane and the sphere). Moreover, in this case the fibres in  $O(\mathcal{M})$  are  $SO(n, \mathbb{R})$  which is a compact group. Below we give the Schmidt metric for the general case of 1 + 1 Riemannian manifolds and compute the curvature scalar for the disc, the sphere and the hyperbolic plane.

In the Riemannian case it is a well know fact that if  $\mathcal{M}$  is a 2-D manifold with a Riemannian metric we can find coordinates  $(v, w)$  which transform locally the line element of the metric to a conformally flat form. Therefore, we have

$$ds^2 = \Omega^2(v, w)(dv^2 + dw^2). \quad (\text{A1})$$

An orthonormal basis is given by the vector fields

$$E_1 = \frac{1}{\Omega} \frac{\partial}{\partial v}, \quad \text{and} \quad (\text{A2})$$

$$E_2 = \frac{1}{\Omega} \frac{\partial}{\partial w}. \quad (\text{A3})$$

Any other orthonormal basis is constructed as a linear combination of (A2) as

$$\tilde{E}_1 = \cos \chi \frac{1}{\Omega} \frac{\partial}{\partial v} + \sin \chi \frac{1}{\Omega} \frac{\partial}{\partial w} \quad \text{and} \quad (\text{A4})$$

$$\tilde{E}_2 = \cos \chi \frac{1}{\Omega} \frac{\partial}{\partial w} + \sin \chi \frac{1}{\Omega} \frac{\partial}{\partial v} \quad (\text{A5})$$

for some  $\chi \in \mathbb{R}$ . Let us notice that the coefficients of basis (A4) with respect to  $\frac{\partial}{\partial v}, \frac{\partial}{\partial w}$  define a unique matrix  $\beta$  and its inverse  $\beta^{-1}$ :

$$\beta = \frac{1}{\Omega} \begin{pmatrix} \cos \chi & -\sin \chi \\ \sin \chi & \cos \chi \end{pmatrix} \quad \text{and} \quad \beta^{-1} = \Omega \begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix}. \quad (\text{A6})$$

Notice the main difference with the Lorentzian case in the definition of the matrix  $\beta$ . The Schmidt metric  $\bar{g}$  on  $O(\mathcal{M})$  is given by

$$\bar{g}(X, Y) : \varpi(X) \cdot \varpi(Y) + \theta(X) \cdot \theta(Y), \quad (\text{A7})$$

for  $X, Y \in TO(\mathcal{M})$ . where

$$\theta(\dot{\gamma}) = \Omega \begin{pmatrix} \dot{v} \cos \chi - \dot{w} \sin \chi \\ \dot{v} \sin \chi + \dot{w} \cos \chi \end{pmatrix} \quad (\text{A8})$$

and

$$\varpi(\dot{\gamma}) = \begin{pmatrix} 0 & -(\dot{\chi} + \frac{1}{\Omega}((\partial_v \Omega)\dot{w} + (\partial_w \Omega)\dot{v})) \\ \dot{\chi} + \frac{1}{\Omega}((\partial_v \Omega)\dot{w} + (\partial_w \Omega)\dot{v}) & 0 \end{pmatrix} \quad (\text{A9})$$

giving the line element for the Schmidt metric:

$$ds^2 = \Omega^2(v, w)(dv^2 + dw^2) + \left( d\chi + \frac{1}{\Omega} \left( \frac{\partial \Omega}{\partial v} dw + \frac{\partial \Omega}{\partial w} dv \right) \right)^2 \quad (\text{A10})$$

### The plane

The euclidean metric on the plane is given by the line element

$$ds^2 = dv^2 + dw^2 \quad (\text{A11})$$

which is characterised by  $R = 0$ .

Then using Eq. (A10) we have that the line element for the corresponding Schmidt metric is

$$ds^2 = dv^2 + dw^2 + d\chi^2 \quad (\text{A12})$$

which is just the flat metric in  $O(\mathcal{M})$  so we have  $\bar{R} = 0$ .

### The sphere

The round metric on the sphere is given by the line element

$$ds^2 = d\theta^2 + \sin^2(\theta)d\varphi^2. \quad (\text{A13})$$

Eq. (A13) can be expressed in terms of isothermal coordinates  $(v, w)$  as

$$ds^2 = \frac{1}{\cosh^2 v}(dv^2 + dw^2). \quad (\text{A14})$$

This metric is characterised by  $R = 1$ .

In a similar manner as we did for the plane metric we use Eq. (A10) to get the line element for the Schmidt metric:

$$ds^2 = \frac{1}{\cosh^2 v}(dv^2 + dw^2) + (d\chi - \tanh(v)dw)^2, \quad (\text{A15})$$

with curvature scalar  $\bar{R} = 3/2$ . Notice that in this case the curvature scalar is positive which for Lorentzian manifolds can not happen as a result of Eq.19.

### The hyperbolic plane

The hyperbolic space metric given in the Poincare-half plane model is given by the line element covering the upper half-plane

$$ds^2 = \frac{1}{w^2}(dv^2 + dw^2), \quad (\text{A16})$$

where  $(v, w) \in \mathbb{R}^2$ . This metric is characterised by  $R = -1$ .

Using Eq. A10 we write the line element for the corresponding Schmidt metric as

$$ds^2 = \frac{1}{w^2}(dv^2 + dw^2) + (d\chi - \frac{1}{w}dv)^2, \quad (\text{A17})$$

with curvature scalar  $\bar{R} = -13/6$ .

## Appendix B: Classification of singularities

The notion of  $b$ -incomplete spaces allows us to describe incomplete curves in manifolds with connections. Our initial motivation to study this, was to develop the language to describe pathologies in the geometry as we approach points that in some sense are “boundary points” of the manifold. In this section we describe how the main manifestation of gravity in GR, the curvature of the manifold, can behave along  $b$ -incomplete curves. This is the scheme proposed by Ellis and Schmidt to classify singularities [21, 22].

Suppose that  $p$  is a point of the  $b$ -boundary  $\partial\mathcal{M}$  of  $(\mathcal{M}, g)$ , then  $p$  is a  $C^r$  ( $r \geq 0$ ) *regular boundary point* if there is a spacetime  $(\mathcal{M}', g')$  which contains  $(\mathcal{M}, g)$  as a sub-manifold and such that the Riemann tensor of  $(\mathcal{M}, g)$  exists and it is  $C^r$  on  $p$ . We call  $(\mathcal{M}', g')$  an *extension* of  $(\mathcal{M}, g)$ . Additionally, we require that there is an open set  $\mathcal{U}$  of  $\mathcal{M}'$  such that  $p$  is in  $\mathcal{M}'$ ;  $p$  is a *singular boundary point* otherwise.

If  $p$  is a singular point then the following scenarios can occur:

- $p$  is a *parallelly propagated (p.p.)* curvature singularity if, for some curve  $\gamma$  with endpoint  $p$ , at least one component of the Riemann tensor, with respect to a parallel propagated orthonormal basis along  $\gamma$ , is not continuous. If the  $r$ -covariant derivative of some component of the Riemann tensor is not continuous, we call the singularity a  $C^r$  p.p. curvature singularity.
- If  $p$  is a singular boundary point and it is not a  $C^r$  p.p. curvature singularity, then it is a  $C^r$  *quasi-regular* singularity. A useful example to visualise such singularities is the singularity one encounters in the apex of a cone or spacetimes containing cosmic strings. In this cases the restriction for extending the spacetime is a topological one.
- If  $p$  is a singular boundary point at the end of the curve  $\gamma$  in which some polynomial constructed from the tensors  $g_{ab}$ ,  $R^a_{bcd}$  and  $r$ -covariant derivatives of  $R^a_{bcd}$  does not behave in a  $C^0$  way, then  $p$  is a  $C^r$  *scalar* singularity. Those singularities are necessarily  $C^r$  curvature singularities.
- If  $p$  is a  $C^r$  curvature singularity, but not a scalar singularity, then it is a  $C^r$  *non-scalar* singularity.

Furthermore, we can classify the curvature singularities in the following subclasses:

- $p$  is a matter singularity if the Ricci tensor diverges at  $p$ .
- $p$  is a conformal singularity if the Weyl tensor diverges at  $p$ .
- $p$  is a divergent singularity if the curvature components are unbounded at  $p$ .
- $p$  is a oscillatory singularity if the curvature components are bounded at  $p$ .

The above list gives a description of gravitational singularities in which the physically interesting cases can be described by the behaviour of the curvature along a curve ending at a point  $p$  in the  $b$ -boundary  $\partial\mathcal{M}$ .

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